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Boundary-layer separation at a free streamline. Part 3. Axisymmetric flow and the flow downstream of separation

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The results of Ackerberg (1970, 1971*a*), on the two-dimensional boundary-layer separation at a sharp trailing edge where a free streamline is attached, are extended to the axisymmetric case, with and without swirl. When the flow has swirl, the boundary-layer swirl velocity close to the wall may be opposite to that in the external flow; this may help explain the 'bathtub vortex paradox' observed by Sibulkin (1962). The solutions for the detached shear layers downstream of separation for two-dimensional and axisymmetric flows, with and without swirl, have been obtained. Some misprints in parts 1 and 2 are corrected in the appendix.

1. Introduction

Ackerberg (1970, 1971*a*), parts 1 and 2, considered the two-dimensional boundary-layer separation which occurs just upstream of a free streamline attached to the sharp trailing edge of a body. Unlike the usual separation problem, in which a region of adverse pressure gradient is followed by a point of zero skin friction, these flows are characterized by an extremely favourable pressure gradient and a positive skin friction proportional to the inverse eighth power of the distance from the edge. This paper is concerned with extending the previous results to include axisymmetric flows, with and without swirl, and to determine the two-dimensional and axisymmetric motion in the detached shear layer downstream of the separation point.

Our results indicate that the boundary-layer flow in the azimuthal plane, with or without swirl, is the same to the first few orders (save for the non-dimensionalization) as the two-dimensional flow discussed in parts 1 and 2. When swirl is present the boundary-layer swirl velocity can be represented by a sum of eigenfunctions and higher-order terms arising from the eigenfunctions. The multiplicative constants appearing in the eigenfunction expansion depend on the radial and swirl velocities upstream in the boundary layer, and the swirl velocity very near the wall may be opposite to that in the outer portion of the boundary layer and external flow. This phenomena of flow reversal has been observed by Sibulkin (1962) and Kelly, Martin & Taylor (1964) in connexion with the bathtub vortex draining through a sharp-edged orifice for unsteady flow, and by Neradka (1969) and Weske (1971) for steady flows. Sibulkin proposed a physical explanation of the flow reversal in terms of an induced circulation opposite to that imposed by the external flow as a result of the turning of the boundary-layer vortex lines as they approach the drain. He apparently did not realize that there exists a pressure gradient singularity due to the sharp-edged orifice, which leads to a singular boundary-layer solution with large transverse velocities of $O(R^{-\frac{1}{2}}x^{-\frac{3}{8}})$ in an inner boundary layer and of $O(R^{-\frac{1}{2}}x^{-\frac{5}{8}})$ in an outer boundary layer; here x is the distance from the edge and R is an appropriate Reynolds number. This serves to intensify the turning of the vortex lines and enhance the flow reversal; thus, the results presented here provide a quantitative basis for the physical argument put forth by Sibulkin.

The other results in this paper deal with the flow downstream of separation, and once again the two-dimensional flow and the flow in the azimuthal plane, with or without swirl, are the same to the first several orders. Two solutions are possible for the first-order flow, one of which predicts backflow along the free streamline. The solution with backflow is rejected because the free streamline will turn into the fluid and this is unlikely for the abrupt separation considered here.

Finally, we show that the velocity transverse to the wall and to the free streamline is not continuous as the edge of the plate is approached from upstream and downstream, and a transition region is necessary to join these flows. The size of the region appears to be of $O(R^{-\frac{4}{9}})$, centred about the edge, and in it a complicated viscous–pressure interaction takes place, which will be discussed in a subsequent paper.

In § 2 the separation problem along the plate is formulated mathematically, and in §§ 3 and 4 inner and outer solutions are obtained using matched asymptotic expansions. These results are discussed in connexion with the bathtub vortex in § 5. Section 6 considers the flow downstream of the separation point in the detached shear layer. Some misprints in parts 1 and 2 are corrected in the appendix.

2. Mathematical formulation

A cylindrical co-ordinate system is chosen, with the origin located at the centre of the circular hole and with the axis of symmetry $\bar{z}$ directed into the fluid away from the free streamline (see figure 1). The plate AS lies in the plane $\bar{z} = 0$ with the separation point at $\bar{r} = a$.† We denote dimensional variables by bars and introduce the non-dimensional variables

$$r = \bar{r}/a, \quad z = \bar{z}R^{\frac{1}{2}}/a, \quad u^* = \bar{u}/U_0, \quad v = \bar{v}R^{\frac{1}{2}}/U_0, \quad w = \bar{w}/W_0, \quad p = \bar{p}/\rho U_0^2. \quad (2.1)$$

Here (u, v, w) are the velocity components in the directions of (r, z, ϕ) increasing, U_0 is the speed $(u^2+v^2)^{\frac{1}{2}}$ in the azimuthal plane and W_0 is the azimuthal velocity (the values at the edge of the boundary (or shear) layer for $r \to 1\pm$), ρ is the fluid density, and $R = \rho U_0 a/\mu$ is the Reynolds number, μ being the viscosity.

† The analysis may be extended without difficulty to the more general case where the tangent to the plate at the separation point intersects the $\bar{r}$ axis at a non-zero angle.

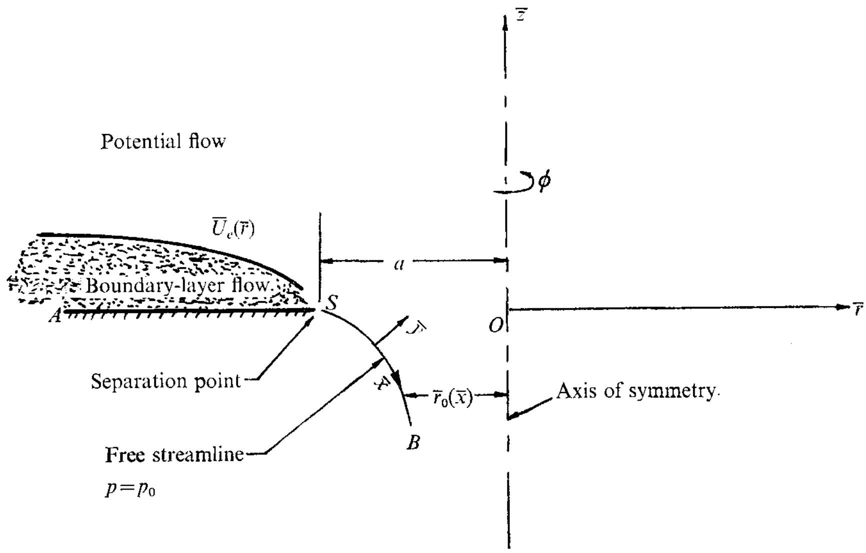

FIGURE 1. Flow geometry.

We assume the motion is axisymmetric and steady, so that $\partial/\partial\phi = \partial/\partial t = 0$. For large values of R, the Prandtl boundary-layer equations may be used provided we delete a small neighbourhood of the edge where a complicated viscous–pressure interaction takes place; these equations are

$$(ru^*)_r + (rv)_z = 0, \tag{2.2}$$

$$u^*u^*_r + vu^*_z - \Gamma r^{-1}w^2 = -p_r + u^*_{zz}, \tag{2.3}$$

$$u^*w_r + vw_z + r^{-1}u^*w = w_{zz}, \tag{2.4}$$

and

$$p_z = o(1), \tag{2.5}$$

where subscripts denote partial differentiation and the parameter $\Gamma = (W_0/U_0)^2$.

Our interest is in the local boundary-layer motion just upstream of the separation point and it is convenient to introduce the new variables

$$x = r-1, \quad Y = z, \quad u(x, Y) = -u^*(r,z), \tag{2.6}$$

and the stream function $\psi(x, Y)$ such that

$$u = (1+x)^{-1}\psi_Y \quad \text{and} \quad v = (1+x)^{-1}\psi_x. \tag{2.7}$$

If (2.6) and (2.7) are substituted in (2.3) and (2.4), we obtain

$$-\psi_{Yx}\psi_Y + \psi_Y^2 + \psi_x\psi_{YY} + \Gamma w^2 = p_x + \psi_{YYY} + O(x), \tag{2.8}$$

and

$$-\psi_Y w_x + \psi_x w_Y - w\psi_Y = w_{YY} + O(x). \tag{2.9}$$

The boundary conditions at the wall and at the edge of the boundary layer are satisfied to a good approximation if

$$\psi = \psi_Y = w = 0 \quad \text{for} \quad Y = 0, \quad x > 0, \tag{2.10}$$

and

$$\psi_Y \to U_e(x), \quad w \to W_e(x) \quad \text{for} \quad Y \to \infty, \quad x > 0, \tag{2.11}$$

where $U_e(x)$ and $W_e(x)$ are the non-dimensional radial and azimuthal velocity components just outside the boundary layer, which are known from the potential-flow solution.

The terminal conditions for $x \to 0$ may be written as

$$\psi_Y \to U_s(Y) \quad \text{and} \quad w \to W_s(Y) \quad \text{for} \quad x \to 0+, \quad Y \geqslant 0, \tag{2.12}$$

where the terminal profiles $U_s(Y)$ and $W_s(Y)$ must be deduced in the course of this analysis.

2.1. *Potential flow*

When there is no swirl ($w \equiv 0$), it was shown by Armstrong (1953) and Ackerberg (1974) that the pressure gradient in the potential flow along the wall just upstream of the separation point has the form

$$p_x = kx^{-\frac{1}{2}} + O(1) \quad (k > 0). \tag{2.13}$$

Moreover, Ackerberg (1974) found (2.13) to be valid not only with swirl ($w \neq 0$) but when there are body forces and a non-zero limit wall curvature for $x \to 0+$. This generalization of (2.13) to include body forces and curvature effects for two-dimensional flow has been rigorously established by Carter (1961), and the two-dimensional results in parts 1 and 2 are valid in these more general situations. For the cases where $k = 0$ in (2.13), the separation will be smooth, but here we are considering only the cases $k > 0$, i.e. abrupt separation.

3. Similarity solution

Since (2.8) exhibits the same singular forcing term (2.13) that arises in the two-dimensional case, it is expected that the velocity field in the azimuthal plane will be similar to the two-dimensional flow. This assumes, of course, that the azimuthal velocity term Γw^2 in (2.8) will not be as singular as $\partial p/\partial x$, a fact which is established *a posteriori*. Therefore, we assume that ψ and w in the inner boundary layer, where Y is small, are of the form

$$\psi^i = 2(\tfrac{1}{2}k)^{\frac{1}{4}} x^{\frac{5}{8}} F(x, \eta), \tag{3.1}$$

$$w^i = (k/\Gamma)^{\frac{1}{2}} x^{\alpha} G(x, \eta). \tag{3.2}$$

The similarity variable η, which will be $O(1)$ in this region, is given by

$$\eta = (\tfrac{1}{2}k)^{\frac{1}{4}}\, Y/x^{\frac{3}{8}}, \tag{3.3}$$

and α is a parameter to be determined. It should be noted when comparing (3.1) with similar expressions for ψ in parts 1 and 2 that we are now using x as independent variable rather than $t = 1 - U_e(x) \propto (-x)^{\frac{1}{2}}$, and the sense of x is reversed. Substituting (3.1) and (3.2) in (2.8) and (2.9), and retaining the largest terms for $x \to 0$, we find

$$F_{\eta\eta\eta} - \tfrac{5}{4} F F_{\eta\eta} + \tfrac{1}{2} F_\eta^2 + 1 - 2x(F_x F_{\eta\eta} - F_\eta F_{x\eta}) + x^{2\alpha + \frac{1}{2}} G^2 + O(x^{\frac{1}{2}}) = 0, \tag{3.4}$$

and

$$G_{\eta\eta} - \tfrac{5}{4} F G_\eta + 2\alpha F_\eta G + 2x(F_\eta G_x - F_x G_\eta) + 2x F_\eta G + O(x) = 0. \tag{3.5}$$

The boundary conditions require

$$F(x,0) = F_\eta(x,0) = G(x,0) = 0, \tag{3.6}$$

and

$F(x,\eta)$ and $G(x,\eta)$ must not contain any exponentially large terms for $\eta \to \infty$. (3.7)

This last condition is necessary for F and G to be matched to the outer boundary-layer solutions, valid for $Y = O(1)$, which will be discussed later.

If $\alpha > -\frac{1}{4}$ and we let $x \to 0$, (3.4) reduces to the equation for $F_0(\eta)$ obtained in parts 1 and 2, and (3.5) yields a linear equation for G with α appearing as an eigenvalue. McLeod (1972) has shown that a solution for F_0 exists, and is unique if it is required to satisfy the additional condition $F_0' \geqslant 0$, which corresponds to no backflow, i.e. the radial velocity at every point is directed toward the free streamline.

In the special case $\alpha = -\frac{1}{4}$, (3.4) and (3.5) yield

$$F''' - \tfrac{5}{4}FF'' + \tfrac{1}{2}F'^2 + 1 + G^2 = 0, \tag{3.8}$$

and

$$G'' - \tfrac{5}{4}FG' - \tfrac{1}{2}F'G = 0, \tag{3.9}$$

subject to the boundary conditions (3.6) and (3.7). Here primes denote differentiation with respect to η. A solution of these equations is $G \equiv 0$, $F = F_0(\eta)$. We now show this is the only acceptable solution.

Assume a second solution (F, G) exists and introduce the integrating factor

$$q(\eta) = \exp\left\{-\frac{5}{4}\int_0^\eta F(\eta)\,d\eta\right\}, \tag{3.10}$$

so that (3.9) may be written in the form

$$(qG')' - \tfrac{1}{2}qF'G = 0. \tag{3.11}$$

We multiply (3.11) by G and integrate over the interval $[0,\infty)$, using integration by parts, to obtain

$$qG'G\Big|_0^\infty - \int_0^\infty qG'^2\,d\eta - \frac{1}{2}\int_0^\infty qF'G^2\,d\eta = 0. \tag{3.12}$$

If we require $F' \geqslant 0$, i.e. no backflow, then $F \geqslant 0$, and $q(\eta)$ will be exponentially small for $\eta \to \infty$. Using the boundary conditions (3.6) and (3.7), it is clear that the boundary term in (3.12) vanishes and we are led to the contradiction

$$0 \leqslant \frac{1}{2}\int_0^\infty qG'^2\,d\eta \Big/ \int_0^\infty qF'G^2\,d\eta = -\tfrac{1}{4}. \tag{3.13}$$

For the cases $\alpha < -\frac{1}{4}$, (3.4) and (3.5) are of no value.

3.1. *The first-order solutions for F and G*

Taking $\alpha > -\frac{1}{4}$, the first-order solution of (3.4) is given by

$$F(x,\eta) = F_0(\eta) + o(1), \tag{3.14}$$

where F_0 is the function, arising from the two-dimensional flow, discussed in parts 1 and 2 and by McLeod (1972). Thus, one of our principal conclusions is that the first-order flow in the azimuthal plane is the same as in the two-dimensional case, and this is independent of the swirl. The lowest-order equation for G will be

$$G_0''' - \tfrac{5}{4}F_0 G_0' + 2\alpha F_0' G_0 = 0, \tag{3.15}$$

subject to

$$G_0(0) = 0, \tag{3.16}$$

and

$G_0(\eta)$ must not contain any exponentially large terms for $\eta \to \infty$. (3.17)

Near $\eta = 0$, two complementary solutions of (3.15) may be found which start with multiples of 1 and η, while for $\eta \to \infty$ there are two (usually different) complementary solutions whose asymptotic expansions start with multiples of

$$y_1(\eta) \sim F_0^{\frac{8\alpha}{5}}, \tag{3.18a}$$

and

$$y_2(\eta) \sim F_0^{-(1+\frac{8}{5}\alpha)} \exp\left\{\int^{\eta} F_0(\eta)\, d\eta\right\}. \tag{3.18b}$$

From part 1, $F_0 \sim A_0^0 \eta^{\frac{5}{3}}$ ($A_0^0 > 0$) and $y_2(\eta)$ will be exponentially large for $\eta \to \infty$; thus to satisfy (3.17), the value of α must be chosen to eliminate any multiple of $y_2(\eta)$. Following the steps that led to (3.12) and (3.13), it is easy to show that

$$\alpha = \frac{1}{2}\int_0^{\infty} q(\eta)\,[G_0'(\eta)]^2\, d\eta \Big/ \int_0^{\infty} q(\eta)\, F_0'(\eta)\,[G_0(\eta)]^2\, d\eta \geqslant 0, \tag{3.19}$$

where $q(\eta)$ is defined in (3.10). Thus, all the eigenvalues are positive. The first three have been found numerically and are

$$\alpha_0 = 0{\cdot}33168\ldots, \quad \alpha_1 = 1{\cdot}3052\ldots, \quad \alpha_2 = 2{\cdot}2831\ldots. \tag{3.20}$$

3.2. *Asymptotic expansions for F and G*

We first note from (3.4) and (3.20) that G will not influence F until terms of $O(x)$ are taken into account. Thus, the first four terms in the expansion for F are similar to those in the two-dimensional theory and we may write

$$F(x,\eta) = F_0(\eta) + Cx^{\frac{1}{2}\gamma}F_1(\eta) + C^2x^{\gamma}F_2(\eta) + C^3x^{\frac{3}{2}\gamma}F_3(\eta) + O(x^{\frac{1}{2}}), \tag{3.21}$$

where $\gamma = 0{\cdot}3157\ldots$ and the functions $F_n(\eta)$ $(n = 1,2,3)$ are the same functions of η, $F_{n\gamma}(\eta)$ $(n = 1,2,3)$, respectively, which have been obtained in part 2. For each value of α, there will be an expansion of the form

$$H_m(x,\eta;\alpha_m) = G_0(\eta) + Cx^{\frac{1}{2}\gamma}G_1(\eta) + C^2x^{\gamma}G_2(\eta) + C^3x^{\frac{3}{2}\gamma}G_3(\eta) + o(x^{\frac{3}{2}\gamma}) \quad (m = 0,1,2,\ldots), \tag{3.22}$$

where the G_n $(n = 1,2,3)$ are solutions of the ordinary differential equations

$$G_1''' - \tfrac{5}{4}F_0G_1' + (2\alpha+\gamma)F_0'G_1 = (\gamma+\tfrac{5}{4})F_1G_0' - 2\alpha F_1'G_0, \tag{3.23}$$

$$G_2''' - \tfrac{5}{4}F_0G_2' + (2\alpha+2\gamma)F_0'G_2 = (2\gamma+\tfrac{5}{4})F_2G_0' - 2\alpha F_2'G_0 + (\gamma+\tfrac{5}{4})F_1G_1' - (\gamma+2\alpha)F_1'G_1, \tag{3.24}$$

$$G_3''' - \tfrac{5}{4}F_0G_3' + (2\alpha+3\gamma)F_0'G_3 = (3\gamma+\tfrac{5}{4})F_3G_0' + (2\gamma+\tfrac{5}{4})F_2G_1' + (\gamma+\tfrac{5}{4})F_1G_2' - 2\alpha F_3'G_0 - (\gamma+2\alpha)F_2'G_1 - (2\gamma+2\alpha)F_1'G_2, \tag{3.25}$$

and each G_n $(n = 1,2,3)$ is subject to the boundary conditions (3.16) and (3.17).

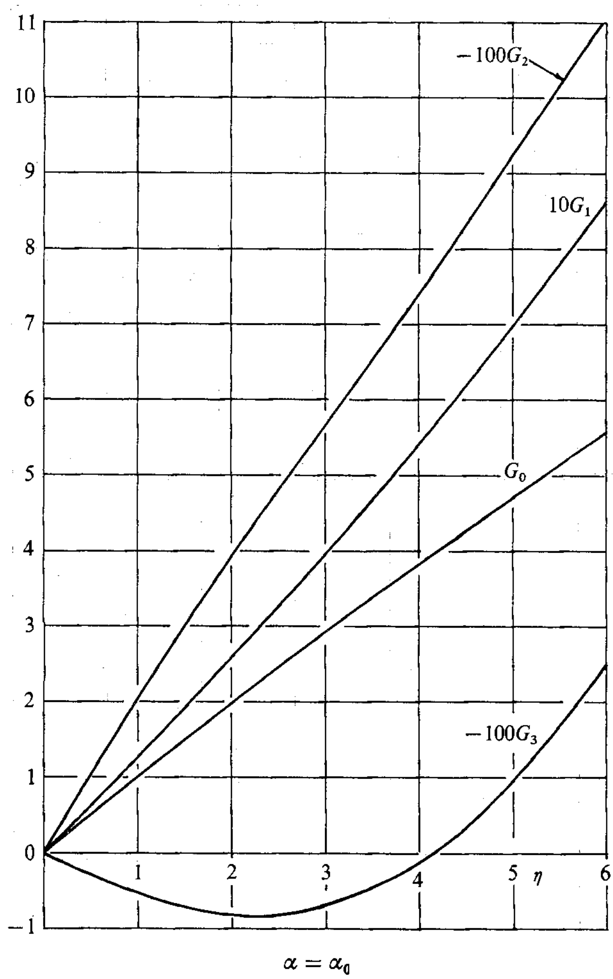

FIGURE 2. $G_0(\eta)$, $G_1(\eta)$, $G_2(\eta)$ and $G_3(\eta)$ against η for $\alpha = \alpha_0$.

	$\alpha_0 = 0{\cdot}33168\ldots$	$\alpha_1 = 1{\cdot}3052\ldots$
$G_0'(0)$	1·00000	1·00000
$G_1'(0)$	0·12948	0·20795
$G_2'(0)$	$-0{\cdot}2116 \times 10^{-1}$	$-0{\cdot}2682 \times 10^{-1}$
$G_3'(0)$	$0{\cdot}5693 \times 10^{-2}$	$0{\cdot}7224 \times 10^{-2}$

TABLE 1

Numerical solutions for $G_n(\eta)$ $(n = 0, 1, 2, 3)$ were obtained for α_0 and α_1 and are displayed in figures 2 and 3. The unrounded values $G_n'(0)$ $(n = 1, 2, 3)$ are given in table 1; here we have used the normalization $G_0'(0) = 1$. The numerical method for finding the eigenvalues consisted of solving the differential equations for $G_0(\eta; \alpha)$ and the variation $V(\eta; \alpha) = \partial G_0/\partial\alpha$ simultaneously. To suppress the exponentially large term (3.18*b*), we integrated backwards from $\eta = 14$ to $\eta = 0$

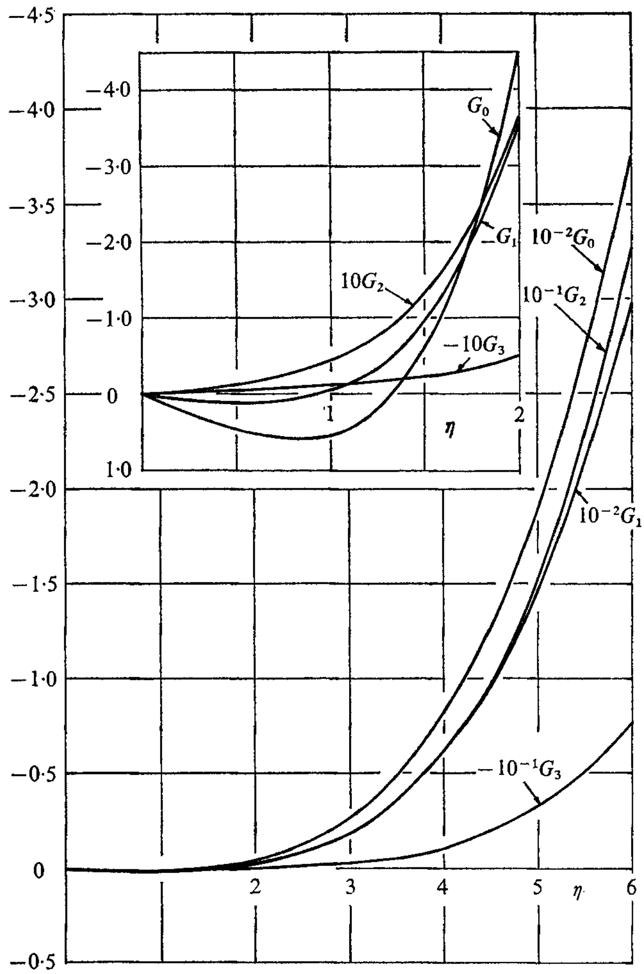

FIGURE 3. $G_0(\eta)$, $G_1(\eta)$, $G_2(\eta)$ and $G_3(\eta)$ against η for $\alpha = \alpha_1$.

and used Newton's method and a Taylor series in α to correct the current value of $\alpha = \alpha_s$ so that $G_0(0;\alpha_s) = 0$, i.e.

$$\alpha_{s+1} = \alpha_s - G_0(0;\alpha_s)/V(0;\alpha_s).$$

Here s denotes the iterate, and α_0 the initial guess. The results converged rapidly and after the third iteration

$$|\alpha_{s+1} - \alpha_s| < 10^{-10}.$$

The computation of the functions $G_n(\eta)$ $(n = 1, 2, 3)$ was made by a method similar to that described in Ackerberg (1971*b*).

3.3. *Asymptotic expansions of $F_n(\eta)$ and $G_n(\eta)$ for $\eta \to \infty$*

In parts 1 and 2 it was found that, for $\eta \to \infty$,

$$F_0(\eta) \sim \eta^{\frac{5}{3}} \sum_{k=0}^{\infty} A_k^0 \eta^{-\frac{1}{3}k}, \tag{3.26}$$

where $A_0^0 = 1{\cdot}950718\ldots$, $A_3^0 = -1{\cdot}577568\ldots$, $A_1^0 = A_2^0 = A_5^0 = 0$, $A_4^0 = 9(5A_0^0)^{-1}$, and

$$F_n(\eta) \sim \eta^{\frac{1}{3}(4n\gamma+5)}(A_0^n + o(1)) + B_n F_0'(\eta) \quad (n = 1, 2, 3), \tag{3.27}$$

where A_0^n, B_n are constants. Using these results in (3.15), (3.23)–(3.25), it is not difficult to show that, for $\eta \to \infty$,

$$G_0(\eta;\alpha) \sim \eta^{\frac{8}{3}\alpha} \sum_{k=0}^{\infty} D_k^0 \eta^{-\frac{1}{3}k}, \tag{3.28}$$

where

$$D_0^0(\alpha_0) = 1{\cdot}197\ldots, \quad D_0^0(\alpha_1) = -0{\cdot}900\ldots, \quad D_1^0 = D_2^0 = D_5^0 = 0,$$
$$D_3^0 = (8A_3^0/5A_0^0)\,\alpha D_0^0, \quad D_4^0 = (72\alpha/25A_0^{02})\,D_0^0, \text{ etc.},$$

$$G_1(\eta;\alpha) \sim \eta^{\frac{4}{3}(2\alpha+\gamma)} \sum_{k=0}^{\infty} D_k^1 \eta^{-\frac{1}{3}k} + \eta^{\frac{1}{3}(8\alpha-3)} \sum_{k=0}^{\infty} \mathscr{D}_k^1 \eta^{-\frac{1}{3}k}, \tag{3.29}$$

and

$$G_n(\eta;\alpha) \sim D_0^n \eta^{\frac{4}{3}(2\alpha+n\gamma)} + \text{smaller terms} \quad (n = 2, 3). \tag{3.30}$$

The values of $D_0^0(\alpha)$ were obtained numerically.

3.4. *Complete inner solution and skin friction*

The complete inner solution for $w(x,\eta)$ will be a sum extending over all the eigenvalues, i.e.

$$w^i(x,\eta) = \sum_{m=0}^{\infty} P_m x^{\alpha_m} H_m(x,\eta;\alpha_m), \tag{3.31}$$

where each term H_m is a sum of the form (3.22) and the P_m are arbitrary constants which depend on the motion in the boundary layer upstream. The value of C in (3.22) is the same for all α_m, and is determined from the solution (3.21).

The radial and azimuthal components of skin friction τ_w^r, τ_w^ϕ, respectively, are given by

$$\tau_w^r = \left.\frac{\partial u}{\partial Y}\right|_{Y=0} = 2\left(\frac{k}{2}\right)^{\frac{3}{4}} x^{-\frac{1}{8}}\,[F_0''(0) + Cx^{\frac{1}{2}\gamma}F_1''(0) + C^2x^{\gamma}F_2''(0) + C^3x^{\frac{3}{2}\gamma}F_3''(0) + O(x^{\frac{1}{2}})], \tag{3.32}$$

where the values $F_n''(0)$ $(n = 0, 1, 2, 3)$ are given in part 2 and

$$\tau_w^\phi = \left.\frac{\partial w}{\partial Y}\right|_{Y=0} = \left(\frac{k}{2}\right)^{\frac{1}{4}} \sum_{m=0}^{\infty} P_m x^{\alpha_m - \frac{3}{8}} \left(\frac{\partial H_m}{\partial \eta}\right)_{\eta=0}. \tag{3.33}$$

If $P_0 \neq 0$, the two largest terms in (3.33) are

$$\tau_w^\phi = c' x^{-0{\cdot}04332\ldots} + c'' x^{0{\cdot}11453\ldots} + \ldots, \tag{3.34}$$

where c' and c'' are constants; thus, the azimuthal skin friction is weakly singular to first order.

4. Principal asymptotic expansion valid for $Y = O(1)$

The similarity solution of § 3 is valid only for small Y, and thus the boundary condition (2.11) has been neglected. We assume the outer solutions, valid for $Y = O(1)$, are of the form (see part 1)

$$\Psi^0 = \Psi_0(Y) + x^{\frac{3}{8}}\Psi_1(Y) + x^{\frac{1}{2}}\Psi_2(Y) + x^{\frac{3}{8}+\frac{1}{2}\gamma}\Psi_3(Y) + x^{\frac{3}{8}+\gamma}\Psi_4(Y) + O(x^{\frac{3}{4}}), \tag{4.1}$$

and

$$W^0 = W_0(Y) + x^{\frac{3}{8}}W_1(Y) + x^{\frac{1}{2}}W_2(Y) + x^{\frac{3}{8}+\frac{1}{2}\gamma} W_3(Y) + x^{\frac{3}{8}+\gamma}W_4(Y) + O(x^{\frac{3}{4}}). \tag{4.2}$$

Once Ψ_0 and W_0 are known, the terminal profiles (2.12) will be given by

$$U_s(Y) = \Psi_0'(Y) \quad \text{and} \quad W_s(Y) = W_0(Y). \tag{4.3}$$

Here primes denote differentiation with respect to Y. The terms in (4.1) and (4.2) of $O[x^{\frac{3}{8}+\frac{1}{2}\gamma}, x^{\frac{3}{8}+\gamma}]$ are required to match with terms arising from the multiples of F_0' in (3.27) and the terms of the second sum in (3.29).

If (4.1) and (4.2) are substituted in (2.8) and (2.9), we obtain a set of ordinary differential equations (in the usual way), which may be integrated to give

$$\Psi_1(Y) = h_1 \Psi_0'(Y), \tag{4.4}$$

$$\Psi_2(Y) = h_2 \Psi_0'(Y) - 2k\Psi_0'(Y) \int^Y [\Psi_0'(s)]^{-2}\, ds, \tag{4.5}$$

$$\Psi_n(Y) = h_n \Psi_0'(Y) \quad (n = 3, 4), \tag{4.6}$$

and

$$W_1(Y) = h_1\, W_0'(Y), \tag{4.7}$$

$$W_2(Y) = h_2\, W_0'(Y) - 2k W_0'(Y) \int^Y [\Psi_0'(s)]^{-2}\, ds, \tag{4.8}$$

$$W_n(Y) = h_n\, W_0'(Y) \quad (n = 3, 4), \tag{4.9}$$

where h_n are constants of integration, and the constant k appears in (2.13).

For $Y \to \infty$, the conditions (2.11) and (2.13) require

$$\Psi_Y^0 \to 1 - 2kx^{\frac{1}{2}} + O(x) \quad \text{for} \quad Y \to \infty, \tag{4.10}$$

and

$$W^0 \to 1 + O(x) \quad \text{for} \quad Y \to \infty, \tag{4.11}$$

where the order of the remainder in (4.11) is known from Ackerberg (1974). If

$$\Psi_0'(Y), W_0(Y) \to 1 + \text{exponentially small terms for } Y \to \infty, \tag{4.12}$$

then

$$\Psi_1' \to 0,\ \Psi_2' \to -2k,\ \Psi_n' \to 0\ (n = 3, 4) \text{ and } W_n \to 0\ (n = 1, 2, 3, 4) \quad \text{for } Y \to \infty;$$

therefore the outer boundary condition (2.11) will be satisfied.

4.1. *The matching requirement and the terminal velocity profiles*

The matching of the inner and outer solutions will determine the form of the terminal velocity profiles (4.3). This requires

$$\lim_{\eta \to \infty} \psi^i(x, \eta),\ w^i(x, \eta) = \lim_{Y \to 0} \Psi^0(x, Y),\ W^0(x, Y). \tag{4.13}$$

The limit on the left is one in which Y is non-zero, fixed but very small, with $x \to 0$. The asymptotic results of § 3.3 are used to evaluate the left-hand side, which is then expressed in terms of x, Y; the result should agree, term-by-term, with the right-hand side of (4.13). The elimination of exponentially large terms in the inner solutions via (3.7) is now seen to be necessary because the terminal profiles Ψ_0', W_0 are expected to behave like powers of Y (and possibly $\ln Y$) for $Y \to 0$.

The terminal profiles are determined from the largest terms in the asymptotic expansions of each F_n and G_n for $\eta \to \infty$. Therefore, using (3.26)–(3.30), we find

$$\Psi_0(Y) = 2\left(\frac{k}{2}\right)^{\frac{2}{3}} Y^{\frac{5}{3}} \sum_{m=0}^{3} A_0^m \left(\frac{k}{2}\right)^{\frac{1}{3}m\gamma} C^m Y^{\frac{4}{3}m\gamma} + O(Y^3), \tag{4.14}$$

and

$$W_0(Y) = \sum_{m=0} P_m \left(\frac{k}{2}\right)^{\frac{2}{3}\alpha_m} Y^{\frac{8}{3}\alpha_m} \sum_{s=0} D_0^s \left(\frac{k}{2}\right)^{\frac{1}{3}s\gamma} C^s Y^{\frac{4}{3}s\gamma}. \tag{4.15}$$

If $P_0 \neq 0$, the first four terms in the terminal velocity profiles are given by

$$U_s(Y) = \omega_0 Y^{\frac{2}{3}} + \omega_1 Y^{\frac{2}{3}(1+2\gamma)} + \omega_2 Y^{\frac{2}{3}(1+4\gamma)} + \omega_3 Y^{\frac{2}{3}(1+6\gamma)} + \ldots, \tag{4.16}$$

$$W_s(Y) = \beta_0 Y^{\frac{8}{3}\alpha_0} + \beta_1 Y^{\frac{4}{3}(2\alpha_0+\gamma)} + \beta_2 Y^{\frac{4}{3}(2\alpha_0+2\gamma)} + \beta_3 Y^{\frac{4}{3}(2\alpha_0+3\gamma)} + \ldots, \tag{4.17}$$

where ω_i and β_i $(i = 0, 1, 2, 3)$ are constants. Some integral powers of Y may appear in the inner sum of (4.15) owing to integral powers in (4.14) which arise from the forcing terms p_x in (2.8).

5. Discussion of results

Our results indicate that the axisymmetric flow in the azimuthal plane, with or without swirl, has the same analytical character as the two-dimensional flow, at least up to terms including the fourth order. In a flow with swirl, it is likely that the first few eigenfunctions $G_0(\eta; \alpha_m)$ will predominate. Some interesting situations might arise if the second eigenfunction were the most important one. From figure 3, we see that an azimuthal flow reversal is likely in the inner boundary layer depending, of course, on the constant C appearing in (3.21) and (3.22). We note that a flow reversal has been observed by Sibulkin (1962) for unsteady swirling flow draining through a sharp-edged orifice when the liquid surface approaches the bottom of the vessel. Sibulkin suggested a physical explanation based on the turning of the boundary-layer vortex lines as they approach the drain. We not only agree with this explanation, but believe that separation at a sharp edge would serve to intensify the flow reversal due to the highly negative axial velocity component near the edge. It should be noticed that even with a multiple of the first eigenfunction present, the function $G_3(\eta; \alpha_0)$ changes sign (see figure 2) and might conceivably help in reversing the azimuthal flow direction in the inner boundary layer.

Attempts were made to duplicate Sibulkin's results by Kelly *et al.* (1964), but they encountered difficulties in reproducing the flow reversals and noted that when the experimental apparatus was shocked, by a sudden blow, pronounced flow reversals could be induced. We do not find these results surprising or at variance with Sibulkin's explanation for the following reason. The azimuthal velocity w depends on two countable sets of constants which arise from the eigenfunctions associated with the radial and azimuthal velocity components (only a single constant C has been displayed in (3.21), but an infinite number would appear if the series were continued). Therefore, depending on the details of the radial and azimuthal velocity profiles upstream, an extremely wide variety of swirl velocities are possible in the inner boundary layer near the separation point, and these might be observed when the liquid surface approaches the bottom of the vessel. The effect of a sudden blow will be to produce a vortex sheet at the wall which corresponds essentially to changing the initial conditions in the inner boundary layer; this might well induce flow reversal.

It should be emphasized that our results apply to steady flows, while Sibulkin's experiments were with unsteady flows. Weske (1971) and Neradka (1969) repeated Sibulkin's experiments using a steady flow produced by maintaining a constant head of fluid in the draining vessel. They observed flow reversal on the free surface when the liquid height was less than 8 mm, but did not discuss the possibility of a flow reversal below the free surface for liquid heights above 8 mm. This is likely to be the case, since Neradka noted that the reversal of the swirl velocity near the bottom of the vessel started at larger radial distances from the orifice than when it first appeared on the free surface. Weske introduced a drop of detergent on the free surface when flow reversal had occurred there and observed that the reversal disappeared immediately. It is not clear, however, that any reversal below the free surface was eliminated by this procedure, and in our opinion it was not. Weske suggests that surface shear stresses play a role in producing the flow reversal and we agree with this; however, we believe the proper interpretation of the surface shear condition is that it corresponds to changing the initial conditions for part of the boundary layer as explained above. It appears that further experiments, in which the upstream flow conditions in the boundary layer can be carefully controlled, are needed.

6. The flow downstream of separation

To study the motion of the detached shear layer downstream of S (see figure 1), we define a co-ordinate system with $\bar{x}$ measuring arc length along the free streamline SB, and $\bar{y}$ perpendicular to it and directed positively into the fluid. We introduce the non-dimensional variables

$$x = \frac{\bar{x}}{a}, \quad Y = \frac{\bar{y}R^{\frac{1}{2}}}{a}, \quad u = \frac{\bar{u}}{U_0}, \quad v = \frac{\bar{v}R^{\frac{1}{2}}}{U_0}, \quad w = \frac{\bar{w}}{W_0}, \quad p = \frac{\bar{p}}{\rho U_0^2}, \quad r_0(x) = a^{-1}\bar{r}_0\left(\frac{\bar{x}}{a}\right), \tag{6.1}$$

where (u, v, w) are the velocity components in the directions (x, y, ϕ) increasing, and $\bar{r}_0(\bar{x})$ is the distance from the axis of symmetry to a point on the free streamline. If we take $r_0 \equiv 1$, the results obtained in this section for the motion in the azimuthal plane, with zero swirl velocity, apply to the two-dimensional motion downstream of separation which arises from the flows considered in parts 1 and 2.

The appropriate boundary-layer equations are given in Rosenhead (1963, p. 418), and may be written as

$$(r_0 u)_x + (r_0 v)_Y = 0, \tag{6.2}$$

$$uu_x + vu_Y - \Gamma r_0^{-1}(dr_0/dx)\, w^2 = u_{YY}, \tag{6.3}$$

$$uw_x + vw_Y + r_0^{-1}(dr_0/dx)\, uw = w_{YY}. \tag{6.4}$$

Here dr_0/dx is minus the cosine of the angle between the free streamline and the r axis, and for $x \to 0+$, $dr_0/dx = -1 + O(x)$. The equation of the free streamline, for $x = O(1)$, is given by potential theory and (6.2)–(6.4) are valid provided $\kappa\delta$ and $(\partial\kappa/\partial\bar{x})\,\delta^2$ are small compared with unity (see Goldstein 1938, p. 128). Here κ is the curvature of the free streamline in the azimuthal plane, which is $O(a^{-1}x^{-\frac{1}{2}})$ for $x \to 0+$, and δ, the boundary-layer thickness, is of $O(aR^{-\frac{1}{2}})$; thus curvature effects are unimportant if $\bar{x} \gg aR^{-\frac{2}{3}}$. We note later that the boundary-layer approximation fails in the larger region $\bar{x} = O(aR^{-\frac{4}{9}})$ owing to a complicated viscous–pressure interaction and the curvature terms are unimportant here in the region $x = O(1)$.

Our interest now is in the local behaviour just downstream of the separation point, i.e. $x \to 0+$. To satisfy (6.2) we introduce the stream function $\psi(x, Y)$ such that

$$u = r_0^{-1}\psi_Y \quad \text{and} \quad v = -r_0^{-1}\psi_x. \tag{6.5}$$

Noting $r_0(x) = 1 - x + O(x^2)$ from free-streamline theory, (6.3) and (6.4) may be written for $x \to 0+$ as

$$\psi_Y\psi_{Yx} + \psi_Y^2 - \psi_x\psi_{YY} + \Gamma w^2 = \psi_{YYY} + O(x), \tag{6.6}$$

and

$$\psi_Y w_x - \psi_x w_Y - w\psi_Y = w_{YY} + O(x). \tag{6.7}$$

Using the boundary-layer approximation, the zero shear condition along the free streamline requires

$$\psi = \psi_{YY} = w_Y = 0 \quad \text{for} \quad Y = 0, \quad x > 0. \tag{6.8}$$

At the edge of the boundary layer, we must satisfy

$$\psi_Y \to 1 + O(x) \quad \text{and} \quad w \to 1 + O(x) \quad \text{for} \quad Y \to \infty.\dagger \tag{6.9}$$

Finally, the initial conditions require

$$\psi_Y \to U_s(Y) \quad \text{and} \quad w \to W_s(Y) \quad \text{for} \quad x \to 0+, \tag{6.10}$$

where $U_s(Y)$ and $W_s(Y)$ are given by (4.16) and (4.17).

6.1. *The inner solution*

The flow field will again be composed of an inner region where the similarity variable $\eta = O(1)$ (in (3.3), x and Y are now to be interpreted by their new definitions) and an outer region where $Y = O(1)$. We can write solutions in the form

$$\psi^i = 2(k/2)^{\frac{1}{4}}\,x^{\frac{3}{4}}f(x,\eta), \tag{6.11}$$

and

$$w^i = (k/\Gamma)^{\frac{1}{2}}\,x^{\alpha}g(x,\eta). \tag{6.12}$$

To satisfy the initial conditions (6.10), α will have to assume the eigenvalues determined from the upstream solution. Substituting (6.11) and (6.12) in (6.6) and (6.7), we obtain the equations

$$f_{\eta\eta\eta} + \tfrac{5}{4}ff_{\eta\eta} - \tfrac{1}{2}f_\eta^2 + 2x(f_xf_{\eta\eta} - f_\eta f_{x\eta}) - x^{2\alpha+\frac{1}{2}}g^2 + O(x) = 0, \tag{6.13}$$

and

$$g_{\eta\eta} + \tfrac{5}{4}fg_\eta - 2\alpha f_\eta g - 2x(f_\eta g_x - f_x g_\eta) + 2xf_\eta g + O(x) = 0. \tag{6.14}$$

† The error estimates in these expressions arise from the approximation $r_0(x) = 1 + O(x)$ (see (6.5)) and from the results $U_e(x) = 1 + O(x)$, $W_e(x) = 1 + O(x)$, for $x \to 0$, obtained in Ackerberg (1974). In special cases, the neglected terms may be $o(x)$.

We assume the expansion

$$f(x,\eta) = f_0(\eta) + Cx^{\frac{1}{2}\gamma} f_1(\eta) + C^2 x^{\gamma} f_2(\eta) + C^3 x^{\frac{3}{2}\gamma} f_3(\eta) + \ldots, \tag{6.15}$$

and for each value of α there will be an expansion of the form

$$h_m(x,\eta;\alpha_m) = g_0(\eta) + Cx^{\frac{1}{2}\gamma} g_1(\eta) + C^2 x^{\gamma} g_2(\eta) + C^3 x^{\frac{3}{2}\gamma} g_3(\eta) + \ldots, \tag{6.16}$$

with

$$w^i(x,\eta) = \sum_{m=0}^{\infty} P_m x^{\alpha_m} h_m(x,\eta;\alpha_m). \tag{6.17}$$

The constants P_m are the same as those appearing in (3.31). Substituting (6.15) and (6.16) in (6.13) and (6.14), we find in the usual way

$$f_0''' + \tfrac{5}{4} f_0 f_0'' - \tfrac{1}{2} f_0'^2 = 0, \tag{6.18}$$

$$f_n''' + \tfrac{5}{4} f_0 f_n'' - (n\gamma + 1) f_0' f_n' + (n\gamma + \tfrac{5}{4}) f_0'' f_n = \mathscr{F}_n \quad (n = 1,2,3), \tag{6.19}$$

where

$$\mathscr{F}_1 = 0, \quad \mathscr{F}_2 = (\gamma + \tfrac{1}{2}) f_1'^2 - (\gamma + \tfrac{5}{4}) f_1 f_1'',$$

$$\mathscr{F}_3 = (1 + 3\gamma) f_1' f_2' - (\gamma + \tfrac{5}{4}) f_1 f_2'' - (2\gamma + \tfrac{5}{4}) f_1'' f_2,$$

and

$$g_n'' + \tfrac{5}{4} f_0 g_n' - (2\alpha + n\gamma) f_0' g_n = \mathscr{G}_n \quad (n = 0,1,2), \tag{6.20}$$

where

$$\mathscr{G}_0 = 0, \quad \mathscr{G}_1 = 2\alpha g_0 f_1' - (\gamma + \tfrac{5}{4}) g_0' f_1,$$

$$\mathscr{G}_2 = 2\alpha g_0 f_2' + (2\alpha + \gamma) g_1 f_1' - (2\gamma + \tfrac{5}{4}) g_0' f_2 - (\gamma + \tfrac{5}{4}) g_1' f_1.$$

The boundary conditions (6.8) require

$$f_n(0) = f_n''(0) = g_n'(0) = 0 \quad (n = 0,1,2,3), \tag{6.21}$$

and to satisfy (6.10), we must have

$$\lim_{\eta \to \infty} f_n / \eta^{\frac{1}{3}(4n\gamma + 5)} = A_0^n \quad (n = 0,1,2,3), \tag{6.22}$$

$$\lim_{\eta \to \infty} g_n / \eta^{\frac{4}{3}(2\alpha + n\gamma)} = D_0^n \quad (n = 0,1,2). \tag{6.23}$$

The boundary condition (6.9) at the edge of the boundary layer will be satisfied by an outer expansion to be discussed later.

The expansion (6.15) and the functions $f_n(\eta)$ are applicable to the two-dimensional flow downstream of free streamline separation discussed in parts 1 and 2 provided the azimuthal plane and $\bar{x}$, $\bar{y}$ (see figure 1) are interpreted in a two-dimensional sense. We should note that the homogeneous equation (6.20) with $n = 0$ has eigensolutions for $\alpha < 0$ (cf. (3.15), (3.19), (6.20)) such that $g_0(\eta;\alpha)$ is exponentially small for $\eta \to \infty$. These solutions are consistent with (6.9) and (6.10) but yield a singular azimuthal velocity on the bounding streamline $\eta = 0$ for $x \to 0+$.

6.2. *The first-order solutions*

The solution of (6.18) with $f_0'(0) > 0$, and (6.20), with $n = 0$ and $\alpha = \alpha_0, \alpha_1$ were obtained numerically and are displayed in figure 4. A second solution with $f_0'(0) < 0$ has been found and is displayed in figure 5 with the corresponding solutions of (6.20) with $n = 0$ and $\alpha = \alpha_0, \alpha_1$. The physical significance of the second solution is in doubt and will be discussed later. We note, however, that, when $f_0'(\eta) > 0$, it may be shown that the solutions $g_0(\eta;\alpha_m)$ remain of one sign,

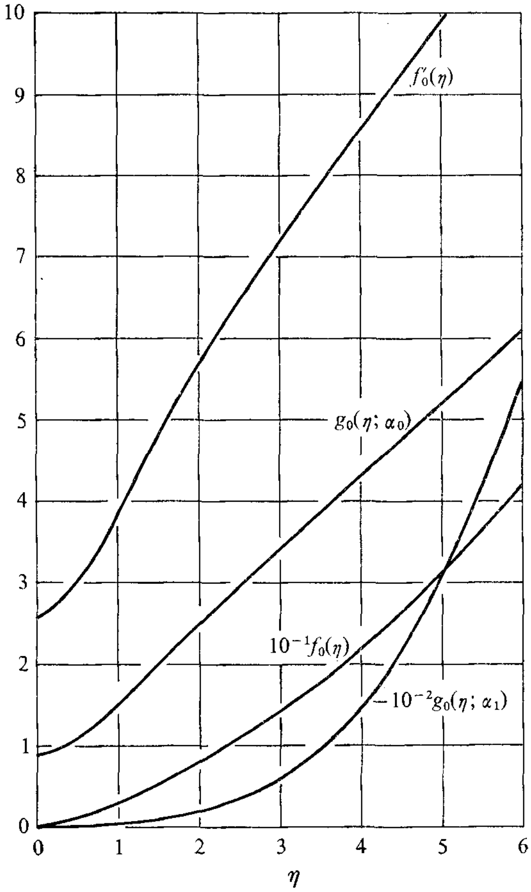

FIGURE 4. $f_0(\eta), f_0'(\eta), g_0(\eta;\alpha_0)$ and $g_0(\eta;\alpha_1)$ against η for $f_0'(0) = 2{\cdot}58491\ldots$.

contrary to the behaviour of $G_0(\eta;\alpha_m)$ upstream. Thus, the possibility of a swirl flow reversal in the free shear layer cannot easily be discerned. The unrounded numerical values for $f_0'(0)$ and $g_0(0;\alpha_m)$ $(m = 0, 1)$ are

$$f_0'(0) = 2{\cdot}58491\ldots,\quad g_0(0;\alpha_0) = 0{\cdot}91003\ldots,\quad g_0(0;\alpha_1) = -0{\cdot}76173\ldots;$$

$$f_0'(0) = -0{\cdot}44869\ldots,\quad g_0(0;\alpha_0) = -0{\cdot}11307\ldots,\quad g_0(0;\alpha_1) = 0{\cdot}024203\ldots.$$

The numerical solutions for f_0 were obtained by choosing the particular values $f'(0) = \pm 1$. In both cases, it was found numerically that $f_0(\eta) \sim D_\pm \eta^{\frac{5}{3}}$ for $\eta \to \infty$ where the constants $D_\pm > 0$. By scaling the dependent and independent variables in (6.18), we find that the solutions satisfying (6.22) will have the initial values

$$f_0'(0) = \pm (A_0^0/D_\pm)^{\frac{3}{4}}, \tag{6.24}$$

where A_0^0 is defined by (3.26). The values $D_\pm$ were obtained from the numerical integrations using the asymptotic expansion given in § 6.3.

The solutions for g_0 were obtained by noting that near $\eta = 0$ two complementary solutions of (6.20) start with multiples of 1 and η, while for $\eta \to \infty$, two

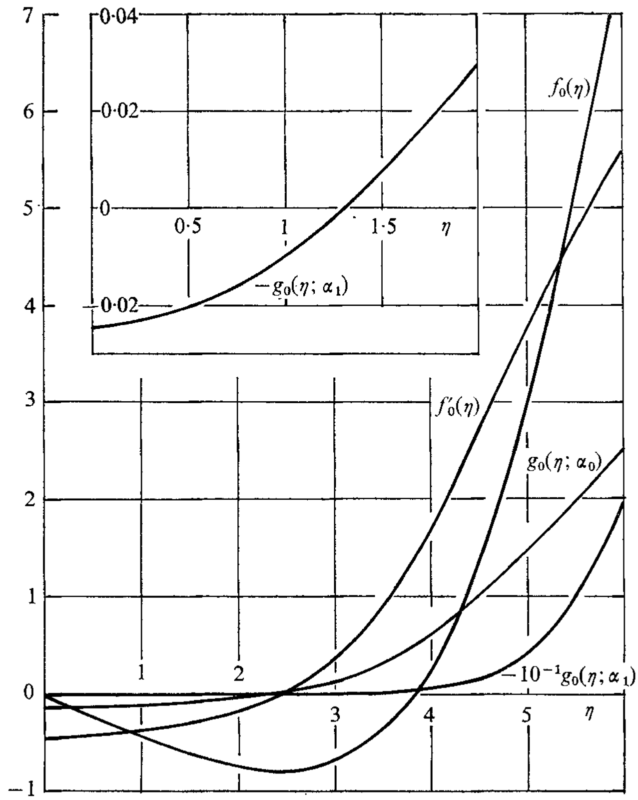

FIGURE 5. $f_0(\eta), f_0'(\eta), g_0(\eta;\alpha_0)$ and $g_0(\eta;\alpha_1)$ against η for $f_0'(0) = -0{\cdot}44869\ldots$.

(usually different) complementary solutions have asymptotic expansions starting with

$$y_1(\eta) \sim f_0^{\frac{8}{5}\alpha}, \tag{6.25}$$

and

$$y_2(\eta) \sim f_0^{-(1+\frac{8}{5}\alpha)} \exp\left\{-\frac{5}{4}\int^\eta f_0(\eta)\,d\eta\right\}. \tag{6.26}$$

To satisfy (6.23), we must choose values of α corresponding to the eigenvalues upstream, and the initial value, $g_0(0,\alpha)$, will fix the constant D_0^0 in (6.23). It is not necessary to suppress an exponentially large term, as in (3.15), and the numerical integration was started at $\eta = 0$ and carried to $\eta = 15$. The multiple of $\eta^{\frac{8}{3}\alpha}$ in the asymptotic expansion of g_0 was determined from the numerical results using the asymptotic expansion in § 6.3.

6.3. *Asymptotic expansions of f_n and g_n for $\eta \to \infty$*

The asymptotic expansion of f_0 for $\eta \to \infty$ is

$$f_0(\eta) \sim \eta^{\frac{5}{3}} \sum_{k=0}^{\infty} a_k^0 \eta^{-\frac{1}{3}k}, \tag{6.27}$$

where

$$a_0^0 = A_0^0 = 1{\cdot}950718\ldots, \quad a_{3+}^0 = 0{\cdot}960298\ldots, \quad a_{3-}^0 = -12{\cdot}108150\ldots,$$

$$a_1^0 = a_2^0 = a_4^0 = a_5^0 = a_7^0 = 0, \quad a_6^0 = (a_3^0)^2/5a_0^0.$$

The values a^0_{3+} and a^0_{3-} correspond to the cases $f'_0(0) \gtrless 0$. Using these results in (6.20) with $n = 0$, we find

$$g_0(\eta;\alpha) \sim \eta^{\frac{8}{3}\alpha} \sum_{k=0}^{\infty} d^0_k \eta^{-\frac{1}{3}k}, \tag{6.28}$$

where

$$d^0_0(\alpha_0) = D^0_0(\alpha_0) = 1\cdot197\ldots, \quad d^0_0(\alpha_1) = D^0_0(\alpha_1) = -0\cdot900\ldots,$$

$$d^0_1 = d^0_2 = d^0_4 = d^0_5 = d^0_7 = 0, \quad d^0_3 = (8a^0_3/5a^0_0)\,\alpha d^0_0, \quad d^0_6 = (2a^0_3/5a^0_0)^2\,\alpha(8\alpha-3)\,d^0_0.$$

The solutions of (6.19) and (6.20), in general, will have asymptotic expansions of the form

$$f_n(\eta) \sim \eta^{\frac{1}{3}(4n\gamma+5)}(a^n_0+o(1))+b_n f'_0(\eta) \quad (n = 1, 2, 3), \tag{6.29}$$

and

$$g_n(\eta) \sim d^n_0\,\eta^{\frac{4}{3}(2\alpha+n\gamma)} + \text{smaller terms} \quad (n = 1, 2, 3). \tag{6.30}$$

The last term of (6.29) arises because $f'_0(\eta)$ is a complementary solution of (6.19). The values $f'_n(0)$, $g_n(0;\alpha)$ are chosen to satisfy (6.22) and (6.23), i.e.

$$a^n_0 = A^n_0 \quad \text{and} \quad d^n_0 = D^n_0. \tag{6.31}$$

6.4. *Downstream principal asymptotic expansion valid for $Y = O(1)$*

To satisfy the boundary conditions (6.9), it is necessary (as in § 4) to find a solution valid for $Y = O(1)$. We assume solutions of the form

$$\psi^0 = \Psi_0(Y)+x^{\frac{3}{8}}\psi_1(Y)+x^{\frac{1}{2}}\psi_2(Y)+x^{\frac{3}{8}+\frac{1}{2}\gamma}\psi_3(Y)+x^{\frac{3}{8}+\gamma}\psi_4(Y)+O(x^{\frac{3}{4}}), \tag{6.32}$$

and

$$w^0 = W_0(Y)+x^{\frac{3}{8}}w_1(Y)+x^{\frac{1}{2}}w_2(Y)+x^{\frac{3}{8}+\frac{1}{2}\gamma}w_3(Y)+x^{\frac{3}{8}+\gamma}w_4(Y)+O(x^{\frac{3}{4}}), \tag{6.33}$$

where Ψ_0 and W_0 are given by (4.14) and (4.15) and Y is to be interpreted by its new definition in § 6. From (4.3), (6.32) and (6.33), we note that (6.10) will be satisfied for $x \to 0+$. If (6.32) and (6.33) are substituted in (6.6) and (6.7), we obtain a set of ordinary differential equations which can be integrated to yield

$$\psi_n(Y) = p_n \Psi'_0(Y) \quad (n = 1, 2, 3, 4), \tag{6.34}$$

and

$$w_n(Y) = p_n W'_0(Y) \quad (n = 1, 2, 3, 4), \tag{6.35}$$

where the p_n are constants of integration. Using the arguments in the discussion following (4.11), it is apparent that the outer boundary condition (6.9) will be fulfilled if Ψ_0 and W_0 satisfy (4.12).

6.5. *The constants h_1 and p_1*

By carrying out the matching described in (4.13), values for the constants h_1 (see (4.4) and (4.7)) and p_1 may be found. Using the notation of (4.14), we find

$$h_1 = \frac{3}{5}\left(\frac{k}{2}\right)^{-\frac{1}{4}}\frac{A^0_3}{A^0_0} \quad \text{and} \quad p^{\pm}_1 = \frac{3}{5}\left(\frac{k}{2}\right)^{-\frac{1}{4}}\frac{a^0_{3\pm}}{A^0_0}. \tag{6.36}$$

The first-order Y velocity component in the region $Y = O(1)$ is given by

$$v(x, Y)\big|_{\text{upstream}} = \tfrac{3}{8}h_1 x^{-\frac{5}{8}}\Psi'_0(Y), \tag{6.37}$$

and

$$v(x, Y)\big|_{\text{downstream}} = -\tfrac{3}{8}p^{\pm}_1 x^{-\frac{5}{8}}\Psi'_0(Y). \tag{6.38}$$

From the numerical results,

$$\tfrac{3}{8}h_1 = \tfrac{9}{40}(\tfrac{1}{2}k)^{-\frac{1}{4}}(-0{\cdot}808716\ldots),$$

$$-\tfrac{3}{8}p_1^+ = \tfrac{9}{40}(\tfrac{1}{2}k)^{-\frac{1}{4}}(-0{\cdot}49227\ldots),$$

$$-\tfrac{3}{8}p_1^- = \tfrac{9}{40}(\tfrac{1}{2}k)^{-\frac{1}{4}}(6{\cdot}2070\ldots).$$

Therefore, as $x \to 0\pm$ for $Y = O(1)$, we find that v is not continuous in any case, and would change sign if the solution corresponding to $f_0'(0) < 0$ were chosen. We think it unlikely that the free streamline will turn toward the interior of the fluid when the separation is abrupt and reject the solution with $f_0'(0) < 0$. There is some experimental evidence indicating that for smooth separation ($k = 0$ in (2.13)) with a free streamline, backflow does occur in some cases and the solution with backflow rejected by Stewartson (1953, p. 568) may be physically relevant. We also note that $\partial w/\partial x$ for $Y = O(1)$ is not continuous for $x \to 0\pm$.

The discontinuity in v, exhibited by (6.37) and (6.38), and in $\partial w/\partial x$ will have to be smoothed out in a transition region near the trailing edge. The size of this region appears to be $|1-r| = O(R^{-\frac{3}{8}})$, and in it a complicated viscous–pressure interaction, similar to that studied by Stewartson (1969) and Messiter (1970), takes place. This will be discussed in a future paper.

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Appendix. Corrections to parts 1 and 2

In part 1, the following equations should read

$$-\frac{1}{\rho}\frac{dp}{dx} = U_e\frac{dU_e}{dx} = \frac{k}{(-x)^{\frac{1}{2}}} - \tfrac{16}{3}k^2 + O(-x)^{\frac{1}{2}}, \tag{3.10}$$

$$\Psi^i(t, Y) = (2^{\frac{1}{2}}k)^{-1}\xi^{\beta}F(\xi,\eta), \tag{4.8}$$

$$u = \xi^{\beta-1}F_\eta, \quad v = (2^{\frac{1}{2}}k)^{-1}\alpha U_e'\xi^{\beta-(1/\alpha)}(\beta F - \eta F_\eta + \xi F_\xi), \tag{4.10}$$

$$\eta^{\frac{2}{3}}, \quad \eta^{\frac{1}{3}(4n+5)}, \quad \eta^{-\frac{1}{3}(4n+14)}\exp\{\tfrac{15}{32}A_0\eta^{\frac{8}{3}}\}, \tag{4.22}$$

$$\tau_w = (\partial u/\partial Y)_{Y=0} = 2^{\frac{1}{2}}kt^{-\frac{1}{4}}[F_0''(0) + c't^{\gamma} + c''t^{2\gamma} + c'''t^{3\gamma} + O(t)], \tag{4.33}$$

$$\tau_w = (2k^3)^{\frac{1}{4}}\,3{\cdot}014015\ldots(-x)^{-\frac{1}{8}} + \ldots. \tag{4.34}$$

Ten lines above (A 1) should read

$$\eta^{\frac{4}{3}(\lambda-1)} \quad \text{or} \quad \eta^{-\frac{1}{3}(4\lambda+7)}\exp\{\tfrac{15}{32}A_0\eta^{\frac{8}{3}}\}.$$

In part 2, the following equations should read

$$\psi(t, Y) = (2^{\frac{1}{2}}k)^{-1} t^{\frac{5}{4}}[F_0(\eta) + Ct^{\gamma}F_{\gamma}(\eta) + C^2t^{2\gamma}F_{2\gamma}(\eta) + C^3t^{3\gamma}F_{3\gamma}(\eta) + tF_1(\eta) + o(t)], \tag{3.1}$$

$$u(t, Y)/U_e = t^{\frac{1}{2}}(1-t)^{-1}[F_0'(\eta) + Ct^{\gamma}F_{\gamma}'(\eta) + C^2t^{2\gamma}F_{2\gamma}'(\eta) + C^3t^{3\gamma}F_{3\gamma}'(\eta) + tF_1'(\eta) + o(t)], \tag{3.9}$$

$$\tau_w = \partial u/\partial Y|_{Y=0} = 2^{\frac{1}{2}}kt^{-\frac{1}{4}}[a_0 + a_{\gamma}Ct^{\gamma} + a_{2\gamma}C^2t^{2\gamma} + a_{3\gamma}C^3t^{3\gamma} + a_1 t + o(t)]. \tag{3.10}$$

In the appendix to part 2 the following equation should read

$$\eta^{\frac{2}{3}}, \quad \eta^{\frac{1}{3}(4\alpha+5)}, \quad \eta^{-\frac{1}{3}(4\alpha+14)}\exp\{\tfrac{15}{32}A_0\eta^{\frac{8}{3}}\}. \tag{A 1}$$

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